Oscillator-Like Hamiltonians and Squeezing

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Generalizing a recent proposal leading to one-parameter families of Hamiltonians and to new sets of *squeezed* states, we construct larger classes of physically admissible Hamiltonians permitting new developments in squeezing. We also discuss coherence.

1. INTRODUCTION

We have recently proposed new sets of Fock states [1] which can be exploited in the contexts of coherence [2] and squeezing [3–5]. Due to the inclusion of a (real, continuous) parameter λ (in the bosonic creation Heisenberg operator), the corresponding oscillator-like "Hamiltonians" led to stationary Schrödinger problems characterized by λ-independent eigenvalues, but λ-*dependent eigenfunctions*, the latter being particularly interesting [1] for the study of new *squeezed* states [3]. Moreover, this approach was in a certain sense a kind of deformation of the current one, but following Wigner's point of view [6].

Let us insist strongly on the fact that we were considering [1] "squeezing" through the λ-dependent eigenfunctions of our Schrödinger problems and, evidently, through the associated mean values of position and momentum, their variances, and (in)equalites coming from the Heisenberg relations. This differs from Yuen's approach [3], which is based on the study of "squeezing" through the famous two-photon coherent states of the radiation field asking for eigenstates of the oscillator operators and not of the Hamiltonian.

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Here we want to generalize such developments by including *a priori* more than one parameter when, simultaneously, we study new creation *and* annihilation operators as well as the corresponding oscillator-like "Hamiltonians" appearing as physically admissible or nonadmissible ones.

The contents are then distributed as follows. In Section 2, we recall a few relations from our first approach [1]. Section 3 is devoted to its already suggested generalization. In Section 4, we apply these considerations to the squeezing problem and find improvements with respect to the previous one-parameter results. Finally some conclusions and comments are included in Section 5.

2. A SHORT SURVEY OF OUR RECENT PROPOSAL

Let us define the new (bosonic) creation Heisenberg operator by

$$a_\lambda^\dagger \equiv a^\dagger + \lambda I, \qquad \lambda \in R \tag{1}$$

where λ refers to a real, continuous parameter and $a^\dagger$ is the Hermitian conjugate of the annihilation operator a satisfying the expected Heisenberg commutation relations, i.e.,

$$[a, a_\lambda^\dagger] = I, \qquad [a, a] = [a_\lambda^\dagger, a_\lambda^\dagger] = 0 \tag{2}$$

These quantum harmonic oscillator-like considerations lead to an analog of a (non-Hermitian) Hamiltonian of the type

$$H_\lambda = \tfrac{1}{2}\{a, a_\lambda^\dagger\} = \tfrac{1}{2}\{a, a^\dagger\} + \lambda a = H_{\text{H.O.}} + \lambda a \tag{3}$$

where the harmonic oscillator Hamiltonian is obviously given by

$$H_{\text{H.O.}} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2, \qquad H_{\text{H.O.}}^\dagger = H_{\text{H.O.}} \tag{4}$$

Moreover, they ensure that

$$[H_\lambda, a] = -a, \qquad [H_\lambda, a_\lambda^\dagger] = a_\lambda^\dagger \tag{5}$$

so that the (generalized) Wigner approach [6] to quantum mechanics is still working. The energy eigenvalues and eigenfunctions have been determined as

$$E_{n,\lambda} = n + \tfrac{1}{2} \qquad (n = 0, 1, 2, \ldots) \tag{6}$$

and

$$\psi_{n,\lambda} = \frac{2^{-n/2}\pi^{-1/4}}{\sqrt{n!}\sqrt{L_n^{(0)}(-\lambda^2)}} e^{-x^2/2} H_n\left(x + \frac{\lambda}{\sqrt{2}}\right) \tag{7}$$

where, as usual, we have chosen units such that $\omega = 1$, $\bar{h} = 1$ and where

H_n and $L_n^{(0)}$ refer to Hermite and generalized Laguerre polynomials [7], respectively. Let us insist on the *unchanged* spectrum (6) with respect to well-known oscillator results, but now with λ-modified eigenfunctions. Moreover, we have shown [1] that these new eigenfunctions (7) correspond to specific *squeezed* states [3, 4]. Let us recall that squeezed states have already been experimentally detected [5] as "two-photon coherent states" for the electromagnetic field. Our new states lead to the characteristic inequality for *squeezing* given by

$$(\Delta x)_\lambda^2 = 2n + \frac{1}{2} - (2\lambda^2 + 1) \frac{L_{n-1}^{(1)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)} - 2\lambda^2 \left(\frac{L_n^{(1)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)} \right)^2 < \frac{1}{2} \quad (8)$$

if

$$n = 1, 2, 3, \ldots \quad \text{and} \quad \lambda \in R \backslash]-r, +r[, \quad r \to 0 \quad \text{if} \quad n \to \infty \quad (9)$$

3. A SIMPLE WAY TO GET GENERALIZED DEVELOPMENTS

The qualities and defects of our above approach [1] suggest the following new approach to the problem: to search for (bosonic) *oscillator-like* annihilation (b) and creation (b^+) operators ensuring the conditions

$$[b, b^+] = 1, \quad [H, b] = -b, \quad [H, b^+] = b^+ \quad (10)$$

and

$$H = \frac{1}{2}\{b, b^+\} = \alpha \frac{d^2}{dx^2} + \beta(x) \frac{d}{dx} + \gamma(x) \quad (11)$$

where b and b^+ have to be general expressions of the usual operators a and $a^\dagger$. Let us note that we have introduced different notations for the usual Hermitian conjugate operator $a^\dagger$ of a and the so-called b^+ associated to b with a general meaning discussed in the following.

Such a set of conditions obviously *contains* the Heisenberg and Wigner requirements through Eqs. (10) and, moreover, restricts the Hamiltonian to Schrödinger-like ones through Eq. (11), where α is a real constant and β, γ are arbitrary real functions of the space variable.

According to such a program, let us introduce the generalized operators b and b^+ in terms of a and $a^\dagger$ by the definitions

$$b = (1 + c_1)a + c_2 a^\dagger + c_3 \quad (12)$$

and

$$b^+ = c_4 a + (1 + c_5)a^\dagger + c_6 \quad (13)$$

where $c_1, c_2, \ldots, c_6$ are arbitrary (real) parameters and where the current

harmonic oscillator context has been included by equating all the parameters to zero, while the definition (1) is also incorporated by equating only c_6 with the λ parameter, all the other c's being identically zero.

At this stage let us point out that the generalized operators (12) and (13) are intimately connected with the construction of the so-called "two-photon coherent states" due to Yuen [3], but here from inhomogeneous linear canonical transformations which could be summarized by the following form:

$$\begin{pmatrix} b \\ b^+ \end{pmatrix} = \begin{pmatrix} 1 + c_1 & c_2 \\ c_4 & 1 + c_5 \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \begin{pmatrix} c_3 \\ c_6 \end{pmatrix}$$

where the matrix has to be invertible so that, for example,

$$(1 + c_1)(1 + c_5) - c_2 c_4 = 1$$

This leads to

$$c_1 + c_5 + c_1 c_5 - c_2 c_4 = 0 \tag{14}$$

which is the constraint between the c's following from (10) and leaving, in fact, only five independent parameters in the whole discussion.

By taking care of the definitions (12) and (13) in the Hamiltonian (11) and by remembering that

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right) \tag{15}$$

we find the possible Hamiltonians in the form

$$H = A\frac{d^2}{dx^2} + (Bx + C)\frac{d}{dx} + Dx^2 + Ex + F \tag{16}$$

transferring the parametrization on the six parameters $A, B, \ldots, F$ given by

$$A = -\tfrac{1}{2} - c_2 c_4 + \tfrac{1}{2}c_4(1 + c_1) + \tfrac{1}{2}c_2(1 + c_5)$$

$$B = c_4(1 + c_1) - c_2(1 + c_5)$$

$$C = \frac{1}{\sqrt{2}}[c_6(c_1 - c_2 + 1) + c_3(c_4 - c_5 - 1)]$$

$$D = \tfrac{1}{2} + c_2 c_4 + \tfrac{1}{2}c_4(1 + c_1) + \tfrac{1}{2}c_2(1 + c_5)$$

$$E = \frac{1}{\sqrt{2}}[c_6(c_1 + c_2 + 1) + c_3(c_4 + c_5 + 1)] \tag{17}$$

$$F = \tfrac{1}{2}c_4(1 + c_1) - \tfrac{1}{2}c_2(1 + c_5) + c_3 c_6$$

These developments compared to the previous ones [1] clearly appear as a

generalization which permits an interesting discussion at the level of physi-
cally admissible Hamiltonians as well as at the level of coherence and (or)
squeezing, once we have solved the eigenvalue and eigenfunction problems
associated with such Hamiltonians.

In terms of the new parameters, let us point out again that the current
harmonic oscillator Hamiltonian (4) corresponds to

$$A = -D = -1/2, \qquad B = C = E = F = 0 \tag{18}$$

while our previous deformation (1) leading to the Hamiltonian (3) is given by

$$A = -D = -1/2, \qquad B = F = 0, \qquad C = E = \lambda/\sqrt{2} \tag{19}$$

With the Hamiltonian (16) and the relations (17), the stationary Schröd-
inger problem can now be solved by conventional quantum mechanical meth-
ods [8]. It leads to the general answer

$$E_n = F - \frac{B}{2} - \frac{C^2}{4A} - \frac{A}{p^2}(2n + 1) + q^2\left(D - \frac{B^2}{4A}\right)$$

$$+ q\left(E - \frac{BC}{2A}\right) \tag{20}$$

while the corresponding eigenfunctions take the form

$$\psi_n(x) = \exp\left[-\frac{B}{4A}x^2 - \frac{C}{2A}x\right]$$

$$\times \exp\left[-\frac{x^2}{2p^2} + \frac{qx}{p^2} - \frac{q^2}{2p^2}\right]H_n\left(\frac{x - q}{p}\right) \tag{21}$$

where p and q enter the necessary change of variable

$$x = py + q \qquad (p \neq 0) \tag{22}$$

Let us mention the two constraints

$$\frac{p^4}{A}\left(D - \frac{B^2}{4A}\right) = -1 \tag{23}$$

and

$$2q\left(D - \frac{B^2}{4A}\right) + E - \frac{BC}{2A} = 0 \tag{24}$$

following from these calculations. Together with Eqs. (17), these relations

(23) and (24) fix the parameters p and q of our change of variable (22) to the unique values

$$p^2 = -2A, \qquad q = 2EA - BC \tag{25}$$

in order to get in particular a positive spectrum. By requiring that we deal with square-integrable eigenfunctions, we finally ask for

$$A < 0 \qquad \text{and} \qquad B < 1 \tag{26}$$

compatible with the specific cases (18) and (19). We thus get (up to a normalization factor N_n) the solutions (21) as given by

$$\psi_n(x) = N_n \exp\left(\frac{1-B}{4A} x^2\right) \exp\left[\left(\frac{(B-1)C}{2A} - E\right)x\right] \exp\left[\frac{(2EA - BC)^2}{4A}\right]$$

$$\times H_n\left(\frac{1}{\sqrt{-2A}} (x - 2EA + BC)\right) \tag{27}$$

Moreover, we obtain the remarkable result of an *unchanged* real spectrum

$$E_n = n + \tfrac{1}{2}, \qquad n = 0, 1, 2, \ldots \tag{28}$$

in this general context as was already the case in our first study [see (6)]. Let us here insist on this real character without having required the self-adjointness of the Hamiltonian (16), a similar property to the one which has recently been quoted by Bender and Boettcher [9], although we have not required any specific discrete symmetries. Nevertheless, we have also noticed that a *necessary* and *sufficient* condition ensuring that $H^\dagger = H$ is simply

$$B = C = 0 \tag{29}$$

leading to a large class of *physically* admissible Hamiltonians.

As a last remark in this section, let us point out that eigenfunctions $\psi_n(x)$ like (27) are once again associated with Fock states—let us call them $|n\rangle_c$, referring to the c-parametrization included in Eqs. (12) and (13)—and it is interesting to quote the action of b and $b^\dagger$ on such states. We obtain

$$b|n\rangle_c = \frac{n}{\sqrt{-A}} \frac{N_n}{N_{n-1}} (1 + c_1 - c_2)|n - 1\rangle_c \tag{30}$$

and

$$b^+|n\rangle_c = \frac{1}{2\sqrt{-A}} \frac{N_n}{N_{n+1}} (1 + c_5 - c_4)|n + 1\rangle_c \tag{31}$$

and point out that

$$bb^+|n\rangle_c = (n + 1)|n\rangle_c, \qquad b^+b|n\rangle_c = n|n\rangle_c \qquad (32)$$

so that the conditions (10) and (11) are obviously satisfied, ensuring in particular that

$$\{b, b^+\}|n\rangle_c = 2H|n\rangle_c = (2n + 1)|n\rangle_c \qquad (33)$$

4. ON IMPLICATIONS IN SQUEEZING

First, we extract new information by considering the lowest energy eigenvalue E_0 of the spectrum and its associated eigenfunction $\psi_0(x)$. Then, we come back very briefly to known cases corresponding to the Hamiltonian (16) with the conditions (18) or (19). Finally, we consider new parametrizations exploiting the results obtained in Section 3 mainly with a view to interesting improvements in squeezing.

4.1. From the Lowest Eigenvalue of the Spectrum

Due to the fundamental and specific role played by the lowest energy eigenvalue E_0, let us study coherence and squeezing through the eigenfunction $\psi_0(x) \equiv (27)$, which takes the explicit form

$$\psi_0(x) = N_0 \exp\left(-\frac{1}{2} \alpha x^2 - \beta x - \frac{1}{2} \gamma\right), \qquad N_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(\frac{\alpha\gamma - \beta^2}{2\alpha}\right)$$

$$(34)$$

in order to ensure that

$$\int_{-\infty}^{+\infty} \psi_0^2(x) \, dx = 1 \qquad (35)$$

where, for brevity, we have introduced the notations

$$\alpha = \frac{B - 1}{2A}, \qquad \beta = E - \frac{(B - 1)C}{2A}, \qquad \gamma = -\frac{1}{2A} (2EA - BC)^2$$

$$(36)$$

In such an $n = 0$ context, the mean values and consequences are readily obtained as follows:

$$\langle x \rangle_0 = \frac{2EA + C}{1 - B}, \qquad \langle x^2 \rangle_0 = \frac{A}{B - 1} + \left(\frac{2EA}{B - 1} - C\right)^2$$

$$\langle p \rangle_0 = 0, \qquad \langle p^2 \rangle_0 = \frac{B - 1}{4A} \qquad (37)$$

so that we get

$$(\Delta x)_0^2 = \frac{A}{B-1}, \qquad (\Delta p)_0^2 = \frac{B-1}{4A} \qquad (38)$$

These results ensure *coherence* due to the Heisenberg relation

$$(\Delta x)_0(\Delta p)_0 = 1/2 \qquad (39)$$

and *squeezing* on the x variable

$$(\Delta x)_0^2 < 1/2 \qquad \text{iff} \quad B < 2A + 1 \qquad (40)$$

or on the p variable

$$(\Delta p)_0^2 < 1/2 \qquad \text{iff} \quad B > 2A + 1 \qquad (41)$$

Such inequalities on A and B only will suggest our future parametrizations in Sections 4.2 and 4.3. In fact, let us immediately note that we plan to privilege the discussion on the x variable, so that Eq. (40) will play the main role.

4.2. From Known Cases . . .

(i) The *harmonic oscillator* context characterized by the condition (18) is well known as far as coherence and squeezing are concerned [2–5]. As already mentioned, this case is contained in our study, but we learn only that it corresponds to all c's equal to zero in Eqs. (12) and (13), it generates a self-adjoint Hamiltonian (4), and deals with Hermitian conjugated operators $b \equiv a$ and $b^\dagger \equiv a^\dagger$ satisfying the condition

$$(b^\dagger)^\dagger = b \qquad (42)$$

(ii) The *deformed* context characterized by the condition (19) has already been discussed in ref. 1: it breaks down the condition (42) *and* the self-adjointness of the Hamiltonian (3), so that physical connections are here questionable, although they correspond to *real* spectra and to new possibilities of squeezing for $n \neq 0$ [1]. Let us point out that the conditions (29) are obviously in contradiction with Eqs. (19) and that, for $n = 0$, the inequalities (40) cannot be satisfied.

4.3. . . . to New Contexts

By keeping the conditions (29) in order to maintain the self-adjointness of the Hamiltonians, we can also require the condition (42). The latter leads to very simple demands of the type

$$c_1 = c_5, \qquad c_2 = c_4, \qquad c_3 = c_6 \qquad (43)$$

so that we then get families of *physically admissible Hamiltonians* which can be further exploited.

(i) Within such conditions, let us go to a one-parameter λ-deformation with, for example, the values

$$c_1 = c_5 = 2/3, \qquad c_2 = c_4 = 4/3, \qquad c_3 = c_6 = \lambda \qquad (44)$$

Such a case corresponds to the parametrization (17) given by

$$A = -1/18, \qquad C = 9/2, \qquad E = 3\sqrt{2}\lambda, \qquad F = \lambda^2, \qquad B = C = 0 \qquad (45)$$

and the eigenvalue and eigenfunction problem can be completely solved. We get the spectrum (28), and the eigenfunctions (27) take the final form

$$\psi_n(x) = N_n \exp\left(-\frac{9}{2}x^2 - \frac{6}{\sqrt{2}}\lambda x - \lambda^2\right) H_n(3x + \sqrt{2}\lambda) \qquad (46)$$

where the normalization factor is in the form

$$N_n = \frac{\sqrt{3}\pi^{-1/4}2^{-n/2}}{\sqrt{n!}} \qquad (47)$$

Mean values and Heisenberg constraints can then be evaluated and we get

$$\langle x \rangle_\lambda = -\frac{\sqrt{2}}{3}\lambda, \qquad \langle x^2 \rangle_\lambda = \frac{1}{9}\left(2\lambda^2 + n + \frac{1}{2}\right) \qquad (48)$$

and

$$\langle p \rangle_\lambda = 0, \qquad \langle p^2 \rangle_\lambda = 9(n + \tfrac{1}{2}) \qquad (49)$$

so that

$$(\Delta x)_\lambda^2 = \tfrac{1}{9}(n + \tfrac{1}{2}), \qquad (\Delta p)_\lambda^2 = 9(n + \tfrac{1}{2}) \qquad (50)$$

leading to

$$(\Delta x)_\lambda (\Delta p)_\lambda = n + \tfrac{1}{2} \qquad (51)$$

This result is analogous to that of the *undeformed* case, but here it permits squeezing (but on *x only*) due to the relations (40) and (50). In fact, such a squeezing can only take place for $n = 0, 1, 2, 3$.

(ii) A final improvement of this example consists in the possible increase of such n values permitting the squeezing and maintaining the nice property (42) and the self-adjointness of H. This can be realized through the new λ-deformation ($\lambda > 0$) characterized by the values

$$c_1 = c_5 = \frac{(\sqrt{\lambda} - 1)^2}{2\sqrt{\lambda}}, \qquad c_2 = c_4 = \frac{\lambda - 1}{2\sqrt{\lambda}}, \qquad c_3 = c_6 = 0 \quad (52)$$

leading to the relations (17) given now in the form

$$A = -\frac{1}{2\lambda}, \qquad D = \frac{\lambda}{2}, \qquad B = C = E = F = 0 \tag{53}$$

satisfying once again the inequalities (40) when $n = 0$.

Here the eigenfunctions are found as

$$\psi_c(x) = N_n \exp\left(-\frac{\lambda}{2}x^2\right)H_n(\sqrt{\lambda}x), \qquad N_n = \frac{\lambda^{1/4}\,\pi^{-1/4}2^{n/2}}{\sqrt{n!}} \tag{54}$$

and we get in correspondence with Eqs. (50)

$$(\Delta x)_\lambda^2 = \frac{1}{\lambda}\left(n + \frac{1}{2}\right), \qquad (\Delta p)_\lambda^2 = \lambda\left(n + \frac{1}{2}\right) \tag{55}$$

We thus notice once more the validity of Eq. (51) ensuring *coherence* for the particular value $n = 0$ only, but *squeezing* (in the x coordinate) for all the values n satisfying the following inequality:

$$\lambda > 2n + 1 > 0 \tag{56}$$

A further interesting property of the above eigenfunctions (54) [and evidently (46)] is that, due to the characteristics of Hermite polynomials [7], these solutions are not only normalized, but are also *orthogonal*, as can be easily established.

If physical applications require a fixed *finite* set of levels in the energy spectrum, we can always choose, due to the inequality (56), our λ parameter in order to guarantee the squeezing up to this n value.

(iii) As a last context, let us relax the condition (42) and the self-adjointness of the Hamiltonian. This corresponds to an extension of the context discussed in ref. 1 and recalled here in Section 4.2(ii). We can choose, for example,

$$b = a + \lambda a^\dagger, \qquad b^+ = a^\dagger \tag{57}$$

corresponding to all the null parameters c except $c_2 = \lambda$ or to

$$A = \tfrac{1}{2}(\lambda - 1), \qquad B = -\lambda, \qquad C = E = 0,$$
$$D = \tfrac{1}{2}(\lambda + 1), \qquad F = -\tfrac{\lambda}{2} \tag{58}$$

ensuring squeezing on x in the $n = 0$ case if $1 > \lambda > 0$. With the spectrum (28), the associated eigenfunctions here take the form

$$\psi_n(x) = N_n \exp\left[-\frac{1}{2}\left(\frac{1+\lambda}{1-\lambda}\right)x^2\right]H_n\left(\frac{x}{\sqrt{1-\lambda}}\right) \tag{59}$$

They are normalizable with

$$N_n = \frac{\pi^{-1/4}}{n!} \left(\frac{1 + \lambda}{1 - \lambda}\right)^{1/4} (1 + \lambda)^{n/2} F_n^{-1/2}(\lambda) \tag{60}$$

but not orthogonal. Depending on the *even* or *odd* character of n, the functions $F_n(\lambda)$ are respectively given by

$$F_n(\lambda) = \sum_{l=0}^{n/2} \frac{2^{2l}\lambda^{n-2l}}{(2l)![(n/2 - l)!]^2} \qquad (n \text{ even}) \tag{61}$$

or

$$F_n(\lambda) = \sum_{l=0}^{(n-1)/2} \frac{2^{2l+1}\lambda^{n-1-2l}}{(2l + 1)![((n - 1)/2 - l)!]^2} \qquad (n \text{ odd}). \tag{62}$$

These functions enter the evaluation of mean values and Heisenberg constraints for each n value. Specific values are of interest in order to learn the general behavior of the corresponding mean values and their consequences, but these are only exercises. Let us just point out that, for $n = 0$, we get

$$\langle x \rangle_\lambda = 0, \qquad \langle x^2 \rangle_\lambda = \frac{1}{2}\left(\frac{1 - \lambda}{1 + \lambda}\right) = (\Delta x)_\lambda^2$$

$$\langle p \rangle_\lambda = 0, \qquad \langle p^2 \rangle_\lambda = \frac{1}{2}\left(\frac{1 + \lambda}{1 - \lambda}\right) = (\Delta p)_\lambda^2 \tag{63}$$

giving us *coherence* due to

$$(\Delta x)_\lambda^2 (\Delta p)_\lambda^2 = 1/4, \qquad \forall \lambda \tag{64}$$

while *squeezing* requires parametrizations according to

$$1 > \lambda > 0 \qquad \text{or} \qquad -1 < \lambda < 0 \tag{65}$$

in the x or p context, respectively. Coherence is then lost if $n \neq 0$, but squeezing can be installed when specific refined inequalities of the type (65) are valid. The upper and lower bounds on these λ values can be determined by entering the results (59)–(62) depending on the n values we are considering.

5. SOME FURTHER CONCLUSIONS AND COMMENTS

Among the above results, let us point out those obtained more particularly in the Section 4.3(ii) leading to an attracting class of one-parameter self-adjoint Hamiltonians

$$H_\lambda = \tfrac{1}{2}\{b_\lambda, b_\lambda^\dagger\} \tag{66}$$

with

$$b_\lambda = \left(1 + \frac{(\sqrt{\lambda} - 1)^2}{2\sqrt{\lambda}}\right)a + \frac{\lambda - 1}{2\sqrt{\lambda}}\, a^\dagger, \qquad b_\lambda^\dagger = b^+, \qquad (b_\lambda^\dagger)^\dagger = b_\lambda \qquad (67)$$

characterized by a deformation parameter $\lambda > 0$ and corresponding to the current harmonic oscillator case when $\lambda = 1$. Appearing nearly as a trivial result, this family opens possible new studies of squeezing through energy eigenfunctions (54) which are not only normalizable, but also orthogonal among themselves. The possible choice $\lambda > 2n + 1$ with a fixed set of energy eigenvalues given by the usual spectrum (28) may offer an interesting connection to possible experimental realizations for oscillator-like systems in order to test and to realize the associated squeezed states.

One further comment is the possible exploitation of our generalized operators b and b^+ by studying more than one (real) parameter in the definitions (12) and (13) as noticed elsewhere [11].

Another point is that the motivations of deforming our (annihilation and creation) oscillator-like operators [as realized in Eqs. (12) and (13)] were intimately connected with a specific mathematical property called "subnormality of operators" [10], a property already exploited in our previous work [1].

As a final comment, let us also recall that our developments could evidently be extended to the fermionic sector as already specified in our first approach [1]. The generalizations (12) and (13) can be realized on *fermionic* annihilation and creation operators and their consequences can then be deduced. Then the superposition of these bosonic and fermionic contexts could be considered in order to go toward supersymmetric developments [12] by including simultaneously specific physical as well as mathematical properties.

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